Problem Set 3

Problems with (HW) are due Tuesday 9/20 at 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 3.1. (HW) Generators for S_n.

- **(a)** Show that S_n is generated by the $n-1$ elements $(1,k)$ for $k=2,\ldots,n$. [Show that you can get an arbitrary transposition by conjugating $(1,k)$ by some $(1,j)$.]
- **(b)** Show that S_n is generated by 2 elements: $(1,2)$ and $(1,2,3,\ldots,n-1,n)$. [Show that you can get all $(1,k)$ from these two using conjugation and then apply the previous exercise.]

Exercises 3.2. Generators for A_n.

- **(a)** Suppose that σ is a k-cycle and τ is an m-cycle and there is exactly one element of $\{1,\ldots,n\}$ that is in the support of both σ and τ. Show that $\sigma\tau$ is a $(k+m-1)$-cycle.
- **(b)** Show that the product of two disjoint transpositions can also be written as the product of two 3-cycles.
- **(c)** (HW) Use part (a) (with $k=m=2$) and part (b) to prove that A_n is generated by 3 cycles.
- **(d)** (HW) Compute $(1,2,a)(1,b,2)$ for a,b distinct and not equal to 1 or 2. . Use the result as motivation to show that the 3-cycles of the form $(1,2,a)$ generate A_n for $n \geq 4$.

Exercises 3.3. Cayley's Theorem

- **(a)** Let $n=5$ and think of $\mathbb{Z}_n$ in the usual way as $\{0,1,2,3,4\}$ with addition modulo n. For each $a \in \mathbb{Z}_n$ write down in tabular form the function on $\mathbb{Z}_n$ defined by addition of a.
- **(b)** Show that part (a) defines a function from $\mathbb{Z}_5$ to S_5, provided you think of S_5 as the group of permutations of $\{0,1,2,3,4\}$. Show that this function is a homomorphism.
- **(c)** Now consider $\mathbb{Z}_2 \times \mathbb{Z}_2$. Enumerate the 4 elements in any way you choose as a_1, a_2, a_3, a_4. For each a_i define a permutation σ_i by $a_i a_1 = a_{\sigma_i(1)}$, $a_i a_2 = a_{\sigma_i(2)}$, $a_i a_3 = a_{\sigma_i(3)}$, $a_i a_4 = a_{\sigma_i(4)}$.
- **(d)** Show in part (c) that this gives a homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to S_4.
- **(e)** (HW) Similarly, the next steps define a homomorphism from D_3 to S_6. Enumerate the elements of as follows $D_3 = \left\{a_1 = r^0, a_2, = r, a_3 = r^2, a_4 = t, a_5 = rt, a_6 = r^2 t\right\}$. For each a_i define a permutation σ_i in S_6. σ_1 is the identity, and σ_2 is given by $\sigma_2(i) = k$ whenever $r a_i = a_k$. Verify that each σ_i is indeed a permutation by writing it in permutation notation.

(f) (HW) Verify in three examples that for any $a, b \in D_3$, the permutation corresponding to ab equals the product of the permutations corresponding to a and b.

(g) (HW) Which elements of D_3 correspond to odd permutations in S_6?

Exercises 3.4.

(a) (HW) Identify all possible signatures for elements of S_5 and the order of these elements. What is the exponent of S_5?

(b) (HW) For each possible signature in S_5, count how many elements have that signature. Check that you get the correct total number of elements in S_5.