Problem Set 4

Problems with (HW) are due Tuesday 9/27 at 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 4.1. (HW) Exercise 2.6.2 The **quaternion group** is defined by

$$Q = \langle a, b \mid a^4 = 1, b^2 = a^2, ba = a^{-1}b \rangle$$

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- **(a)** Show that Q has 8 elements. List them in a useful fashion and show how to multiply them as we did for the dihedral group.
- **(b)** Show that Q has 1 element of order 2 and 6 of order 4.
- **(c)** Draw the lattice diagram for this group.

Exercises 4.2. Exercise 2.7.10

- **(a)** Show that A_n is invariant under conjugation: for any $\pi \in S_n$, $\pi A_n \pi^{-1} = A_n$.
- **(b)** Let C_n be the rotation subgroup of D_n. Find two elements of C_4 that are conjugate as elements of D_4 but are not conjugate as elements of C_4.
- **(c)** (HW) Find two elements of D_4 that are conjugate as elements of S_4 but are not conjugate as elements of D_4. A computer algebra system will be useful.
- **(d)** Consider D_n as a subset of S_n by enumerating the vertices of an n-gon clockwise $1, 2, \ldots, n$. Show that the n-cycle $(1, 2, \ldots, n)$ and any reflection generate D_n.

Exercises 4.3. (HW)

- **(a)** Let $\sigma \in S_n$. Let $(a_1, a_2, \ldots, a_k) \in S_n$ be a k-cycle, so the a_i are distinct. Show that
$$\sigma * (a_1, a_2, \ldots, a_k) * \sigma^{-1} = \Big(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)\Big) =$$
[Consider two cases, $b = \sigma(a_i)$ for some i, and $b \notin \{\sigma(a_1), \sigma(a_2), \ldots \sigma(a_k)\}$.]

Exercises 4.4. Recall that the exponent of a group G is the lcm of the orders of the elements (if this is finite).

- **(a)** For a finite group G show that the the exponent of G divides the order of G.
- **(b)** Give an example to show that there may not be an element in G whose order is the exponent of G.

Exercises 4.5. (HW) Define a function $\varphi_a : G \longrightarrow G$ by $\varphi(g) = aga^{-1}$.

- **(a)** Show that φ_a is an automorphism of G.
- **(b)** Show that $\varphi : G \longrightarrow \mathrm{Aut}(G)$ defined by $\varphi : a \longmapsto \varphi_a$ is a homomorphism. The image, $\{\varphi_a : a \in G\}$, is therefore a subgroup of $\mathrm{Aut}(G)$. It is called $\mathrm{Inn}(G)$, the group of **inner automorphisms** of G.
- **(c)** What is the kernel of φ?

Exercises 4.6.

(a) Let N be a normal subgroup of G. For any subgroup H of G, $H \cap N$ is a normal subgroup of H.

(b) (HW) If $\varphi : G \longrightarrow H$ is a homomorphism and N is normal in H, then $\varphi^{-1}(N)$ is normal in G.

(c) (HW) The **center** of G is the set of elements in G that commute with all elements of G, $Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$. Any subgroup of the center of G, including $Z(G)$ itself, is normal in G.

(d) (HW) Find all normal subgroups of D_4 and D_5.

(e) Show that any subgroup of index 2 is normal.