Problem Set 5

Problems with (HW) are due Tuesday 10/4 at 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 5.1. 2.11.5. Some normal subgroups

(a) Show that the intersection of two normal subgroups of G is normal in G.

(b) Let G be a group, possibly infinite. Let I be some indexing set and for each $i \in I$ let H_i be a subgroup of G. Prove that for any $a \in G$,

$$a\Big(\bigcap_{i\in I} H_i\Big)a^{-1} = \bigcap_{i\in I} aH_ia^{-1}$$

(c) (HW) Let H be a subgroup of G and let $N = \bigcap_{g\in G} g^{-1}Hg$. Prove that N is normal in G.

(d) (HW) Let $n \in \mathbb{N}$ and let K be the intersection of all subgroups of G of order n. Prove that K is normal in G.

Exercises 5.2. 2.9.12

(a) Show that the subgroup of upper triangular 2×2 matrices is conjugate to the group of lower triangular matrices. [Hint: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.]

(b) Show that the set of matrices with nonzero determinant of the form $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$ is a coset of the upper triangular matrices.

Exercises 5.3. 2.9.15

(a) Show that D_4 is isomorphic to the matrix group with elements $\{\pm\mathbf{1}, \mathbf{r}, \mathbf{s}, \mathbf{t}\}$ where

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) Draw the lattice diagram for this matrix group (it looks just like D_4, but use the elements here).

(c) (HW) More generally find a subgroup of $\mathrm{GL}_2(\mathbb{R})$ that is isomorphic to D_n. (Remember your trigonometry.)

Exercises 5.4. Let $H = H(F)$ be the set of 3 by 3 upper triangular matrices over a field F with 1s on the diagonal.

(a) Give a brief explanation of why this is indeed a subgroup of $\mathrm{GL}(3, F)$.

(b) Show that the following 3 types of matrices generate this group.

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Let $F = \mathbb{F}_p$. Explain why H is then generated by 3 matrices, those in the form above with $a = b = c = 1$.

(d) Show that $H(\mathbb{F}_2) \cong D_4$.

(e) (HW) Show that the center $Z(H)$ consists of all matrices of the form $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Furthermore $Z(H) \cong (F, +)$.

(f) (HW) Show that $H(\mathbb{F}_p)/Z(G)$ is isomorphic to $F \times F$.

(g) (HW) Conclude that G is metabelian.

Exercises 5.5. 2.11.6. The normalizer and centralizer of a subgroup. Let K be a subgroup of G and define

$$N_G(K) = \{g \in G : gKg^{-1} = K\}$$
$$C_G(K) = \{g \in G : gkg^{-1} = k \text{ for all } k \in K\}$$

These are called the **normalizer** of K in G and the **centralizer** of K in G.

(a) Show that $N_G(K)$ is a subgroup of G.

(b) Show that K is a normal subgroup of $N_G(K)$.

(c) (HW) If $H \leq G$ and K is a normal subgroup of H show that $H \leq N_G(H)$. So, $N_G(K)$ is the largest subgroup of G in which K is normal.

(d) (HW) Show that $C_G(K)$ is a normal subgroup of $N_G(K)$.

(e) (HW) Show that $N_G(K)/C_G(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$.

Exercises 5.6. (Challenge Problem) 2.11.7The commutator subgroup. In a group G, the **commutator** of a, b is $aba^{-1}b^{-1}$. Notice that this is e_G iff a and b commute. The **commutator subgroup** of a group G is the group G' generated by the commutators.

$$G' = \langle aba^{-}b^{-1} : a, b \in G \rangle$$

(a) Compute the commutator subgroup of D_n (two cases: n odd and n even). Think of D_n as generated by r, t with $r^n = t^2 = e$ and $tr = r^{n-1}t$.

(b) Write down the commutator of the conjugation of a by x and the conjugation of b by x.

(c) Prove that G' is a normal subgroup of G. It is enough to show that the conjugation of any commutator is another commutator.

(d) Prove that G/G' is abelian.

(e) Prove that G/N abelian implies $G' \leq N$. So, the commutator subgroup of G is the smallest normal subgroup N group such that the quotient G/N is abelian.