Problem Set 6

Problems with (HW) are due Thursday 10/13 at 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 6.1. Let N, H be two groups and let $\varphi : H \longrightarrow \mathrm{Aut}(N)$ be a homomorphism. Write $\varphi(h)$ as φ_h. Define the **external semi-direct product** of N and H determined by φ to be the set $N \times H$ with multiplication

$$(n_1, h_1) * (n_2, h_2) = (n_1\varphi_{h_1}(n_2), h_1h_2)$$

This is written $N \rtimes_\varphi H$. Show that this does indeed define a group.

(a) Verify that (e_H, e_K) is the identity element.

(b) Show that each element does have an inverse.

(c) (HW) Show that the associative law holds.

Exercises 6.2. Verify the following are semi-direct products and connect each construction with the definition of external semi-direct product from problem 6.1.

(a) $D_n \cong \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ where $\varphi : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ takes the non-identity element of $\mathbb{Z}_2$ to the automorphism of $\mathbb{Z}_n$ taking a to $-a$.

(b) $S_n = A_n \rtimes \langle(1, 2)\rangle$. What is the map φ?

(c) (HW) $S_4 = V \rtimes S_3$ where V is the Klein-4 subgroup

$$V = \{\mathrm{id}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$$

What is the map φ?

(d) (HW) In $\mathrm{GL}_n(F)$, for F a field, let T be the upper triangular matrices with nonzeros on the diagonal; let U be the upper triangular matrices with 1's on the diagonal and let D be the diagonal matrices with nonzero elements on the diagonal. For $n = 2$, show that $T = U \rtimes D$. Describe the map $\varphi : D \longrightarrow \mathrm{Aut}(U)$.

(e) (Challenge) Do the previous problem for arbitrary n.

Exercises 6.3. Let F be a field. Let $\mathrm{GL}_n(F))$ be the general linear group: $n \times n$ matrices over F with nonzero determinant. Let $\mathrm{SL}_n(F))$ be the special linear group: matrices with determinant 1. Let F^*I be the nonzero multiples of the identity matrix. In this problem we investigate the finite fields F and values of n for which $\mathrm{GL}_n(F) \cong \mathrm{SL}_n(F) \times F^*I$.

(a) Show that $\mathrm{SL}_n(F)$ and F^*I are both normal in $\mathrm{GL}_n(F)$.

(b) Show that $|\mathrm{GL}_n(F)| = |\mathrm{SL}_n(F)||F * I_n|$.

(c) (HW) For the fields $F = \mathbb{F}_3$ and $F = \mathbb{F}_5$, show that $\mathrm{GL}_n(F)$ is a direct product as above for n odd, but not for n even.

(d) (HW) For the field $F = \mathbb{F}_7$, show that $\mathrm{GL}_n(F)$ is a direct product as above for n coprime to 6, and is not otherwise.

(e) (Challenge) For which fields $\mathbb{F}_q$ and which n is $\mathrm{GL}_n(\mathbb{F}_q)$ a direct product as above?

Exercises 6.4.

(a) (HW) Use the definition of external semi-direct product to create the other non-abelian group of order 12 (besides D_6 and A_4), $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ where φ is the only possible map $\mathbb{Z}_4 \longrightarrow \mathrm{Aut}(\mathbb{Z}_3)$ that is not trivial. Let a be the generator for $\mathbb{Z}_3$ and b the generator for $\mathbb{Z}_4$. Show the following:

(1) Every element can be represented uniquely as $a^i b^j$ for $i \in \{0,1,2\}$ and $b \in \{0,1,2,3\}$

(2) The group can be presented as $\langle a, b | a^3 = b^4 = 1, ba = a^2 b\rangle$

(3) Find the inverse of $a^i b^j$.

(4) Find a general formula for $a^i b^j * a^m b^n$. You can break this into cases if you want.

(b) (HW) Use the definition of external semi-direct product to create the only non-abelian group of order 21 (the smallest non-abelian group of odd order), $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Let a be the generator for $\mathbb{Z}_7$ and b the generator for $\mathbb{Z}_3$. Show how to represent, invert, and multiply elements of this group as you did in the previous problem.

(c) (Challenge Problem) Use the definition of external semi-direct product to construct semi-direct products $\mathbb{Z}_m \rtimes \mathbb{Z}_n$. You will need to start with a homomorphism $\varphi : \mathbb{Z}_n \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$. See how many of the small non-abelian groups you can find in the table of small abelian groups on Wikipedia.