Problem Set 9 supplement

Exercises 9.1. Consider $\mathbb{F}_2[x,y]$.

(a) Identify all the elements of $\mathbb{F}_2[x,y]/\langle x^3, y^2\rangle$. How many are there?
(b) Find the nilpotents, zero divisors, and units in $\mathbb{F}_2[x,y]/\langle x^3, y^2\rangle$.
(c) Identify all the elements of $\mathbb{F}_2[x,y]/\langle x^3 y^2\rangle$.
(d) Find the nilpotents, zero divisors, and units in $\mathbb{F}_2[x,y]/\langle x^3 y^2\rangle$.
(e) Find the maximal ideals in $\mathbb{F}_2[x,y]/\langle x^3 y^2\rangle$.

Theorem (4.5.11)**.** *Let R be a ring and I and ideal in R.*

- *I is a maximal ideal if and only if R/I is a field.*
- *I is a prime ideal if and only if R/I is an integral domain.*
- *I is a radical ideal if and only if R/I is reduced.*

Exercises 9.2. In the notes I prove the forward direction.

(a) Prove that R/I is reduced (no nilpotents) implies I is radical.
(b) Prove that R/I is an integral domain implies I is prime.
(c) Prove that R/I is a field implies I is maximal.

Exercises 9.3. Consider now $\mathbb{Q}[x,y]$.

(a) Show that $\langle y + x^2\rangle$ is a prime ideal in $\mathbb{Q}[x,y]$.
(b) Show that for any $a \in \mathbb{Q}$, $\langle x - a, y - a^2\rangle$ is a maximal ideal containing $\langle y - x^2\rangle$.
(c) Show that $\langle x^2 + 1, y + 1\rangle$ is a maximal ideal containing $\langle y - x^2\rangle$. Can you identify other such ideals?