Problem Set 9

Problems with (HW) are due Thursday 11/10 at 11:00 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Exercises 9.1. Let R and S be rings and consider $R \times S$.

- **(a)** Let I be an ideal in R and J and ideal in S. Show that $I \times J$ is an ideal in $R \times S$.
- **(b)** Show that all ideals in $R \times S$ are of the form $I \times J$.

Exercises 9.2. (HW) Let S be a subring of R and J an ideal in R. Show that

- **(a)** $S + J$ is a subring of R.
- **(b)** $S \cap J$ is an ideal in S.

Exercises 9.3. Let $\varphi : R \longrightarrow S$ be a homomorphism.

- **(a)** Show that $\varphi^{-1}(J)$ is an ideal in R for any ideal J in S.
- **(b)** (HW) For I and ideal in R show that $\varphi^{-1}(\varphi(I)) = I + K$ where $K = \ker \varphi$. In particular, if I contains K, then $\varphi^{-1}\varphi(I) = I$.

Exercises 9.4.

- **(a)** Show that the nonzero prime ideals in $\mathbb{Z}$ are also maximal ideals. [Suppose p is a prime number. Try to enlarge $\langle p \rangle$ and show that you get all of $\mathbb{Z}$.]
- **(b)** (HW) Let I be a nonzero proper ideal in $\mathbb{Z}$. We know I is principle; let a be the smallest positive integer in I. Show that I is radical if and only if the prime factorization of a is $a = p_1 p_2 \cdots p_r$ for *distinct* primes p_i.
- **(c)** (HW) Extend these results to $F[x]$ for F a field.

Exercises 9.5. (HW) Let $\varphi : R \longrightarrow S$ be a homomorphism of rings and let J be an ideal in S. From an exercise above, we know that $\varphi^{-1}(J)$ is an ideal in R.

- **(a)** If J is a radical ideal, show that $\varphi^{-1}(J)$ is a radical ideal in R.
- **(b)** If J is a prime ideal, show that $\varphi^{-1}(J)$ is a prime ideal in R.
- **(c)** Using $R = \mathbb{Z}$ and $S = \mathbb{Q}$ show that $\varphi^{-1}(J)$ may not be maximal when J is maximal.

Exercises 9.6. (HW)

- **(a)** Show that the intersection of two radical ideals is radical.
- **(b)** Illustrate with an example from $F[x]$ for F a field.
- **(c)** Given an example in $F[x]$ to show that the intersection of two prime ideals may not be prime.

Exercises 9.7.

(a) Let $N = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ be the set containing 0 and all of the nilpotent elements in a ring R. Show that N is an ideal of R. It is called the **nilradical** of R.

(b) (HW) Show that N is contained in the intersection of all prime ideals in R.

(c) (HW) Show that if $a \in N$ then $1 + a$ and $1 - a$ are units.

(d) (Challenge) Show N equals the intersection of all prime ideals in R.