Problem Set 4

Problems here may be on the test.

Problems 4.1. Recall the lattice for $\mathbb{Z}_2 \times \mathbb{Z}_4$. (Or $\mathbb{Z}_4 \times \mathbb{Z}_4$ for a bigger challenge.)

(1) For each subgroup, H of $\mathbb{Z}_2 \times \mathbb{Z}_4$, find the lattice for the quotient group $\mathbb{Z}_2 \times \mathbb{Z}_4 / H$

(2) Is $\mathbb{Z}_2 \times \mathbb{Z}_4 / H$ cyclic? Find generator(s) for it.

Problems 4.2. Conjugation in S_n.

(1) Let $\sigma \in S_n$. Let $(a_1, a_2, \ldots, a_k) \in S_n$ be a k-cycle, so the a_i are distinct. Show that

$$\sigma * (a_1, a_2, \ldots, a_k) * \sigma^{-1} = \Big(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)\Big)$$

[Consider two cases, $b = \sigma(a_i)$ for some i, and $b \notin \{\sigma(a_1), \sigma(a_2), \ldots \sigma(a_k)\}$. Explain why this breakdown into two cases makes sense.]

(2) Let π and σ be elements of S_n. Show that the signature of $\sigma\pi\sigma^{-1}$ is the same as the signature of $\pi \in S_n$.

Problems 4.3. More examples of conjugation.

(1) Show that A_n is invariant under conjugation: for any $\pi \in S_n$, $\pi A_n \pi^{-1} = A_n$.

(2) Let C_n be the rotation subgroup of D_n. Find two elements of C_4 that are conjugate as elements of D_4 but are not conjugate as elements of C_4.

(3) Find two elements of D_4 that are conjugate as elements of S_4 but are not conjugate as elements of D_4. A computer algebra system will be useful.

(4) Consider D_n as a subset of S_n by enumerating the vertices of an n-gon clockwise $1, 2, \ldots, n$. Show that the n-cycle $(1, 2, \ldots, n)$ and any reflection generate D_n.

Problems 4.4. For a an element of a group G, define a function $\varphi_a : G \longrightarrow G$ by $\varphi_a(g) = aga^{-1}$.

(1) Show that φ_a is an automorphism of G.

(2) Show that $\varphi : G \longrightarrow \text{Aut}(G)$ defined by $\varphi : a \longmapsto \varphi_a$ is a homomorphism. The image, $\{\varphi_a : a \in G\}$, is therefore a subgroup of $\text{Aut}(G)$. It is called $\text{Inn}(G)$, the group of **inner automorphisms** of G.

(3) What is the kernel of φ?

Problems 4.5. The **quaternion group** is defined by

$$Q = \langle a, b \mid a^4 = 1, b^2 = a^2, ba = a^{-1}b \rangle$$

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(1) Show that Q has 8 elements. List them in a useful fashion and show how to multiply them as we did for the dihedral group.

(2) Show that Q has 1 element of order 2 and 6 of order 4.

(3) Draw the lattice diagram for this group.

Problems 4.6. Recall that the exponent of a group G is the lcm of the orders of the elements (if this is finite).

(1) For a finite group G show that the the exponent of G divides the order of G (Lagrange).

(2) Give an example to show that there may not be an element in G whose order is the exponent of G.