Problem Set 5

Problems with (HW) are due Wednesday 10/11 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Problems 5.1. Some normal subgroups.

(1) Show that the intersection of two normal subgroups of G is normal in G.

(2) Let G be a group, possibly infinite. Let I be some indexing set and for each $i \in I$ let H_i be a subgroup of G. Prove that for any $a \in G$,

$$a\Big(\bigcap_{i\in I} H_i\Big)a^{-1} = \bigcap_{i\in I} aH_ia^{-1}$$

[Does this element-wise; if g is in the LHS, show it is in the RHS, then vice-versa.]

(3) (HW) Let H be a subgroup of G and let $N = \bigcap_{g\in G} g^{-1}Hg$. [Use the result from (2).] Prove that N is normal in G.

(4) (HW) Let $n \in \mathbb{N}$ and let K be the intersection of all subgroups of G of order n. Prove that K is normal in G. [Use the result from (2).]

Problems 5.2. Let N, H be two groups and let $\varphi : H \longrightarrow \mathrm{Aut}(N)$ be a homomorphism. Write $\varphi(h)$ as φ_h. Define the **external semi-direct product** of N and H determined by φ to be the set $N \times H$ with multiplication

$$(n_1, h_1) * (n_2, h_2) = (n_1\varphi_{h_1}(n_2), h_1h_2)$$

This is written $N \rtimes_\varphi H$. Show that this does indeed define a group.

(1) Verify that (e_H, e_K) is the identity element.

(2) Show that each element does have an inverse.

(3) (HW) Show that the associative law holds.

Problems 5.3. This problem shows the relationship between internal semi-direct products and the construction of external semi-direct products.

(1) Show $D_n \cong \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ where $\varphi : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ takes the non-identity element of $\mathbb{Z}_2$ to the automorphism of $\mathbb{Z}_n$ taking a to $-a$.

(2) $S_n = A_n \rtimes \langle(1, 2)\rangle$. What is the map φ?

(3) (HW) $S_4 = V \rtimes S_3$ where V is the Klein-4 subgroup

$$V = \{\mathrm{id}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$$

What is the map φ?