Problem Set 6

Problems with (HW) are due Wednesday 10/18 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Problems 6.1.

(1) Show that the subgroup of upper triangular 2×2 matrices is conjugate to the group of lower triangular matrices. [Hint: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.]

(2) Show that the set of matrices with nonzero determinant of the form $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$ is a coset of the upper triangular matrices.

Problems 6.2.

(1) Show that D_4 is isomorphic to the matrix group over $\mathbb{Q}$ with elements $\{\pm\mathbf{1}, \pm\mathbf{r}, \pm\mathbf{s}, \pm\mathbf{t}\}$ where

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(2) Draw the lattice diagram for this matrix group (it looks just like D_4, but use the elements here).

(3) More generally find a subgroup of $\mathrm{GL}_2(\mathbb{R})$ that is isomorphic to D_n. (Remember your trigonometry.)

Problems 6.3. Let $H = H(F)$ be the set of 3 by 3 upper triangular matrices over a field F with 1s on the diagonal.

(1) Give a brief explanation of why this is indeed a subgroup of $\mathrm{GL}(3, F)$.

(2) Show that the following 3 types of matrices generate this group.

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

(3) Let $F = \mathbb{F}_p$. Explain why H is then generated by 3 matrices, those in the form above with $a = b = c = 1$.

(4) Show that $H(\mathbb{F}_2) \cong D_4$.

(5) (HW) Show that the center $Z(H)$ consists of all matrices of the form $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Furthermore $Z(H) \cong (F, +)$.

(6) (HW) Show that $H/Z(H)$ is isomorphic to $F \times F$.

Problems 6.4. In $\mathrm{GL}_n(F)$, for F a field, let T be the upper triangular matrices with nonzeros on the diagonal; let U be the upper triangular matrices with 1's on the diagonal and let D be the diagonal matrices with nonzero elements on the diagonal.

(1) (HW) For $n = 2$, show that $T = U \rtimes D$. Describe the map $\varphi : D \longrightarrow \mathrm{Aut}(U)$.

(2) (Challenge) Do the previous problem for arbitrary n.

Problems 6.5. Let F be a field. Let $\mathrm{GL}_n(F))$ be the general linear group: $n \times n$ matrices over F with nonzero determinant. Let $\mathrm{SL}_n(F))$ be the special linear group: matrices with determinant 1. Let F^*I be the nonzero multiples of the identity matrix. In this problem we investigate the finite fields F and values of n for which $\mathrm{GL}_n(F) \cong \mathrm{SL}_n(F) \times F^*I$.

(1) Show that $\mathrm{SL}_n(F)$ and F^*I are both normal in $\mathrm{GL}_n(F)$.

(2) Show that $|\mathrm{GL}_n(F)| = |\mathrm{SL}_n(F)||F^*I_n|$.

(3) (HW) For the fields $F = \mathbb{F}_3$ and $F = \mathbb{F}_5$, show that $\mathrm{GL}_n(F)$ is a direct product as above for n odd, but not for n even.

(4) (HW) For the field $F = \mathbb{F}_7$, show that $\mathrm{GL}_n(F)$ is a direct product as above for n coprime to 6, and is not otherwise.

(5) (Challenge) For which fields $\mathbb{F}_q$ and which n is $\mathrm{GL}_n(\mathbb{F}_q)$ a direct product as above?

Problems 6.6. Using the definition of external semi-direct product.

(1) Create the other nonabelian group of order 12 (besides D_6 and A_4), $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ where φ is the only possible map $\mathbb{Z}_4 \longrightarrow \mathrm{Aut}(\mathbb{Z}_3)$ that is not trivial.

(1) (HW) Let a be the generator for $\mathbb{Z}_3$ and b the generator for $\mathbb{Z}_4$. Show that every element can be represented uniquely as $a^i b^j$ for $i \in \{0, 1, 2\}$ and $b \in \{0, 1, 2, 3\}$

(2) The group can be presented as $\langle a, b | a^3 = b^4 = 1, ba = a^2 b\rangle$

(3) Find the inverse of $a^i b^j$.

(4) Find a general formula for $a^i b^j * a^m b^n$. You can break this into cases if you want.

(2) (HW) Use the definition of external semi-direct product to create the only non-abelian group of order 21 (the smallest non-abelian group of odd order), $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Let a be the generator for $\mathbb{Z}_7$ and b the generator for $\mathbb{Z}_3$. Show how to represent, invert, and multiply elements of this group as you did in the previous problem.

(3) (Challenge Problem) Use the definition of external semi-direct product to construct semi-direct products $\mathbb{Z}_m \rtimes \mathbb{Z}_n$. You will need to start with a homomorphism $\varphi : \mathbb{Z}_n \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$. See how many of the small non-abelian groups you can find in the table of small abelian groups on Wikipedia.