Problem Set 7

We will discuss these in class on Monday 10/23 and they may be on the test. Please attempt them before class so that the time in class is most productive.

Problems 7.1.

(1) Find the elementary divisors and the invariant factors for $\mathbb{Z}/30 \times \mathbb{Z}/600 \times \mathbb{Z}/420$.

(2) Find the elementary divisors and the invariant factors for $\mathbb{Z}/50 \times \mathbb{Z}/75 \times \mathbb{Z}/136 \times \mathbb{Z}/21000$.

(3) Let $n = p^6 q^5 r^4$ where p, q, r are distinct primes. How many abelian groups are there of order n?

(4) For n as in (3), how many abelian groups of order n have k invariant factors, for $k = 1, 2, 3, 4, 5, 6$? Check that that the total of these values is the same as the response to the previous question.

Problems 7.2. The torsion subgroup of an abelian group. Let A be an infinite abelian group. Let $\mathrm{Tor}(A)$ be the set of elements with finite order, which is called the **torsion subgroup** of A.

(1) Show that $\mathrm{Tor}(A)$ is, indeed, a normal subgroup of A.

(2) Show that $\mathrm{Tor}(A) = \bigcup_{m \in \mathbb{N}} A[m]$. (Note that, even inside an abelian group, the union of subgroups is not usually a group!)

(3) Show that $\mathrm{Tor}(A/\mathrm{Tor}(A))$ is trivial. That is, letting $T = \mathrm{Tor}(A)$, the only element of finite order in A/T is the identity element, $e + T$.

(4) Give an example (a simple one!) of a finitely generated abelian group in which the identity element together with the elements of infinite order do *not* form a subgroup. (As opposed to the torsion subgroup.)

Problems 7.3.

(1) Let A be an abelian group. Suppose $f : A \to \mathbb{Z}$ is a surjective homomorphism with kernel K. Show that A has an element a such that A is the internal direct product $K \times \langle a \rangle$.

(2) In the previous problem, suppose f is not surjective but $f(A) = n\mathbb{Z}$ for some $n \in \mathbb{N}$. Show that it still holds that there is an element $a \in A$ such that A is the internal direct product $K \times \langle a \rangle$.

Problems 7.4. Infinitely generated abelian groups can be more complicated than finite ones. Consider the group $\mathbb{Q}/\mathbb{Z}$.

(1) On a number line, sketch a region that contains exactly one element for each equivalence class of $\mathbb{Q}/\mathbb{Z}$.

(2) Show that for any integer n there is an element of order n in $\mathbb{Q}/\mathbb{Z}$.

(3) How many elements of order n are there in $\mathbb{Q}/\mathbb{Z}$?

(4) Show that every element has finite order.

(5) Show that every nontrivial cyclic subgroup is generated by $\frac{1}{n}$ for some integer $n > 1$.

(6) Show that $\mathbb{Q}/\mathbb{Z}$ is not finitely generated as an abelian group.

(7) (Challenge) Show that $\mathbb{Q}/\mathbb{Z}$ cannot be written as a direct product of $\langle a \rangle$ and another group H for any nonzero $a \in \mathbb{Q}/\mathbb{Z}$.

Problems 7.5. This problem fleshes out some details in the proof of Theorem 3.3.8: A_n is simple for $n > 5$.

(1) Within the alternating group A_n for each $i = 1, \ldots, n$, let $G_i = \{\sigma \in A_n : \sigma(i) = i\}$. Show that G_i is a subgroup of A_n.

(2) Find a $\pi \in A_n$ such that each $G_i = \pi G_j \pi^{-1}$.

(3) Justify the statement in the fourth paragraph of the proof of 3.3.8 "Then $(\tau\sigma\tau^{-1})\sigma^{-1} \in N$ and straightforward computation shows $\tau\sigma\tau^{-1}\sigma^{-1}(c) = c$."

(4) Justify the statement in the fifth paragraph of the proof of 3.3.8 "$(\tau\sigma\tau^{-1})\sigma^{-1} \in N$ and $\tau\sigma\tau^{-1}\sigma^{-1}(b) = b$."