Problem Set 9

Problems with (HW) are due 11/13 in class. Your homework should be easily legible, but need not be typed in Latex. Use full sentences to explain your solutions, but try to be concise as well. Think of your audience as other students in the class.

Problems 9.1. Let R and S be rings and consider $R \times S$.

(1) Let I be an ideal in R and J an ideal in S. Show that $I \times J$ is an ideal in $R \times S$.

(2) Show that all ideals in $R \times S$ are of the form $I \times J$.

Problems 9.2. Let $\varphi : R \longrightarrow S$ be a homomorphism.

(1) Show that $\varphi^{-1}(J)$ is an ideal in R for any ideal J in S.

(2) For I and ideal in R it may not be the case that $\varphi(I)$ is and ideal in S. Give an example.

(3) (HW) For I an ideal in R show that $\varphi^{-1}(\varphi(I)) = I + K$ where $K = \ker \varphi$. In particular, if I contains K, then $\varphi^{-1}\varphi(I) = I$.

Problems 9.3. Let $N = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ be the set of the nilpotent elements in a ring R. It is called the **nilradical** of R.

(1) Show that N is an ideal of R.

(2) (HW) Show that N is contained in the intersection of all prime ideals in R.

(3) (HW) Show that if $a \in N$ then $1 + a$ and $1 - a$ are units.

Problems 9.4. (HW) Let $\varphi : R \longrightarrow S$ be a homomorphism of rings and let J be an ideal in S. From an exercise above, we know that $\varphi^{-1}(J)$ is an ideal in R.

(1) If J is a radical ideal, show that $\varphi^{-1}(J)$ is a radical ideal in R.

(2) If J is a prime ideal, show that $\varphi^{-1}(J)$ is a prime ideal in R.

(3) Using $R = \mathbb{Z}$ and $S = \mathbb{Q}$ show that $\varphi^{-1}(J)$ may not be maximal when J is maximal.

Problems 9.5. Maximal, prime and radical ideals in $\mathbb{Z}$ and $F[x]$.

(1) Show that the nonzero prime ideals in $\mathbb{Z}$ are also maximal ideals. [Suppose p is a prime number. Try to enlarge $\langle p \rangle$ and show that you get all of $\mathbb{Z}$.]

(2) Let I be a nonzero proper ideal in $\mathbb{Z}$. We know I is principle; let a be the smallest positive integer in I. Show that I is radical if and only if the prime factorization of a is $a = p_1 p_2 \cdots p_r$ for *distinct* primes p_i.

(3) Extend these results to $F[x]$ for F a field.

Problems 9.6.

(1) Show that the intersection of two radical ideals is radical.

(2) Illustrate with an example from $F[x]$ for F a field.

(3) Given an example in $F[x]$ to show that the intersection of two prime ideals may not be prime.

Problems 9.7. Consider $\mathbb{F}_2[x,y]$.

(1) Identify all the elements of $\mathbb{F}_2[x,y]/\langle x^3, y^2\rangle$. How many are there?
(2) Find the nilpotents, zero divisors, and units in $\mathbb{F}_2[x,y]/\langle x^3, y^2\rangle$.
(3) Identify all the elements of $\mathbb{F}_2[x,y]/\langle x^3 y^2\rangle$. How would you write down a general element?
(4) Find the nilpotents, zero divisors, and units in $\mathbb{F}_2[x,y]/\langle x^3 y^2\rangle$.
(5) Find all maximal ideals in $\mathbb{F}_2[x,y]/\langle x^3 y^2\rangle$.

Theorem (4.6.11)**.** *Let R be a ring and I and ideal in R .*

- *I is a maximal ideal if and only if R/I is a field.*
- *I is a prime ideal if and only if R/I is an integral domain.*
- *I is a radical ideal if and only if R/I is reduced.*

Problems 9.8. In the notes I prove the forward direction.

(1) Prove that R/I is reduced (no nilpotents) implies I is radical.
(2) Prove that R/I is an integral domain implies I is prime.
(3) Prove that R/I is a field implies I is maximal.