Linear Algebra
Math 254
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Topics for Final Exam
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Solving Linear Systems

- Transform a system of linear equations into a matrix equation.
- Solve a system using Gaussian elimination.
- Explain the steps that you use (switching rows, scaling a row, adding a row to another one).
- Identify the matrix operation corresponding to each step.
- Write a vector x as a sum of vectors $v_1, \ldots, v_m$.
- Find the RREF form of a matrix.
- Find a basis for the nullspace of a matrix.
- Find a basis for the column space of a matrix.
- Find a basis for the row space of a matrix.
- Invert a matrix using Gaussian elimination.

Vector Space Terminology

- Definitions you should know:
  - linear combination, span, linearly independent, basis;
  - subspace, linear transformation;
  - rank, nullspace (a.k.a. kernel), column space(aka image) of a matrix;
  - orthogonal, orthonormal basis,
  - eigenvalue, eigenvector, eigenspace.
- Decide whether a function is a linear transformation.
- Be able to state and use the Rank Theorem (Sec. 4.6 Thm 14).
- Know the equivalences in the Inverse Matrix Theorems (Sec. 2.3 Thm 8, Sec 4.6 p. 267).

Orthogonality

- In $\mathbb{R}^2$, be able to identify all orthonormal bases.
- Find the components of a vector relative to a subspace.
  Given $V \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, decompose x, relative to V, as $x = \hat{x} + z$ with $\hat{x} \in V$ and $z \in V^{\perp}$.
- Find the components of $x \in V$ relative to an orthonormal basis of V.
- Transform a given basis to an orthonormal basis using Gram-Schmidt.
- Write the QR decomposition of a matrix.

Determinants

- Know the basic properties of determinants (Sec. 3.2).
- Similar matrices have the same determinant.
- Compute an arbitrary 2×2 or 3×3 determinant.
- Compute the determinant of larger matrices with special conditions (e.g. lots of zeros).

Eigenvalues and Eigenvectors

- Find the characteristic polynomial of an $n \times n$ matrix A.
- Find the eigenvalues for 2×2, and (doable) 3×3 matrices and triangular matrices.
- Find the algebraic multiplicity and the geometric multiplicity of each eigenvalue.
- Find a basis for the eigenspace associated to each eigenvalue.
- Diagonalize a matrix A when it is possible.
- Understand diagonalization as change of basis.
- Determine the matrix for a population change problem as Sec. 4.9 and Ch. 5.
- Compute long term behavior of a discrete dynamical system using eigenvalues and eigenvectors.

Here are a few exercises.

1. Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$.

   (a) Write $y = \begin{bmatrix} 2 \\ -3 \\ -10 \end{bmatrix}$. as a linear combination of the v_i. Use Gaussian elimination and identify each step as a matrix multiplication.
   (b) Find the kernel and image of the transformation $T(x) = Ax$.
   (c) What are the nullity and the rank of A.
   (d) Find the inverse of A.
   (e) Solve part (a) using A^{-1}.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 3 & 1 \\ 4 & 8 & 5 & 3 \end{bmatrix}$$

   (1) Find a basis for the kernel of find a basis for the image of the transformation $T(x) = Ax$.
   (b) What are the nullity and the rank of A?

3. Classify the geometrical properties of the following matrices (look at what each does to the standard basis).

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

4. Find the QR factorization of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

5. Compute the determinant.

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

6. Diagonalize each matrix if possible

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$