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Math 623: Matrix Analysis
Final Exam Preparation

The final exam has two parts, which I intend to each take one hour. Part one is on the material covered since the last exam: Determinants; normal, Hermitian and positive definite matrices; positive matrices and Perron's theorem. The problems will all be very similar to the ones in my notes, or in the accompanying list.

For part two, you will write an essay on what I see as the fundamental theme of the course, the four equivalence relations on matrices: matrix equivalence, similarity, unitary equivalence and unitary similarity (See p. 41 in Horn 2nd Ed. He calls matrix equivalence simply "equivalence."). Imagine you are writing for a fellow master's student. Your goal is to explain to them the key ideas.

Matrix equivalence:

(1) Define it.

(2) Explain the relationship with abstract linear transformations and change of basis.

(3) We found a simple set of representatives for the equivalence classes: identify them and sketch the proof.

Similarity:

(1) Define it.

(2) Explain the relationship with abstract linear transformations and change of basis.

(3) We found a set of representatives for the equivalence classes using Jordan matrices. State the Jordan theorem.

(4) The proof of the Jordan theorem can be broken into two parts: (1) writing the ambient space as a direct sum of generalized eigenspaces, (2) classifying nilpotent matrices. Sketch the proof of each. Explain the role of invariant spaces.

Unitary equivalence

(1) Define it.

(2) Explain the relationship with abstract inner product spaces and change of basis.

(3) We found a simple set of representatives for the equivalence classes: identify them.

(4) Discuss the singular value decomposition and its proof, and polar decomposition.

Unitary similarity:

(1) Define it.

(2) Explain the relationship with abstract inner product spaces and change of basis.

(3) We don't have a neat general set of representatives for equivalence classes under unitary similarity. If you want you can summarize the short portion of my notes [Normal §3] on Sprecht's theorem.

(4) We do have strong results for certain special families of matrices. State the equivalence classes for normal matrices, for Hermitian matrices, and for unitary matrices. Discuss these and related results and proofs.

In your essay you should do items (1)-(3) for each equivalence relation, spending roughly 25 minutes total. You should then spend the remainder of the time on the more complicated material in items (4). For this portion you may choose to focus on whatever you wish: in depth presentation of one equivalence relation, survey/comparison of more than one.

- You will not be able to use your notes or a book.
- There is no "best answer." I want to see your individual perspective.
- Your choice of presentation is based on what you think is important, interesting, and possible given your understanding and the time limitation.
- Some positive attributes that I will look for:
  - Your mathematical grammar is correct.
  - Your statement of definitions and theorems is correct.
  - You use good examples to illustrate.
  - You give a good proof of some particular result.
  - You give a good high level analysis of the strategy for proving a complex result.
  - You show some creativity in presentation.

(1) Write A^{-1} as a polynomial in A, do not attempt to compute it.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2) Let $A \in M_4$ such that $A^4 = I_4$. show that A is diagonalizable.

(3) What are necessarily the entries of A if it is normal and nilpotent.

(4) Show that any Hermitian matrix $H \in \mathcal{M}_n(\mathbb{C})$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ satisfies:

$a) \frac{x^\star Ax}{x^\star x} \leq \lambda_n, \forall x \in \mathbb{C}^n, x \neq 0.$

$b) \max_{x\neq 0} \frac{x^\star Ax}{x^\star x} = \lambda_n.$

(5) Let K a Hermitian square matrix of size n with at least one eigenvalue positive and at least one eigenvalue negative.

(a) Show that there is a nonzero x such that $x^\star Kx = 0$.
Hint: Let w_i an orthonormal basis for C^n made of eigenvectors of K, and $x = \sum \alpha_i w_i$. Write $x^\star Kx$ in term of this basis and eigenvalues. How could you choose $\alpha_i, i \in \overline{1,n}$?

(b) Can $(x,y)_K = y^\star Kx$ define an inner product on C^n?

(6) Let H be a positive definite matrix. Prove that H is invertible.

(7) Let $A = (a_{ij})$ with $a_{ij} = \min\{i,j\}$. Then A is positive definite.

(8) Let $A = (a_{ij})$ be a positive definite matrix and $B = (b_{ij})$ with

$$b_{ij} = \frac{a_{ij}}{(a_{ii}a_{jj})^{1/2}}.$$

Then B is positive definite, with $b_{ii} = 1$ and $|b_{ij}| < 1$.

(9) What are the singular values of a unitary matrix?

(10) Let $A \in \mathcal{M}_n(\mathbb{C})$ be normal with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that the singular values of A are $|\lambda_1|, \ldots, |\lambda_n|$.

(11) Let $1, 1, 2, 2, 5$ be eigenvalues of $A \in \mathcal{M}_5(\mathbb{C})$. Suppose A is normal. Check true or false and give a brief but complete argument for your choice in the following assumptions. A must be

(a) unitary
(b) positive definite
(c) hermitian
(d) skew-hermitian
(e) unitarily diagonalizable

(12) Let $1, 1, -1, -1, i, i$ be eigenvalues of $A \in \mathcal{M}_6(\mathbb{C})$. Suppose that A is unitarily diagonalizable.

(a) Is A
  i. hermitian?
  ii. skew-hermitian?
  iii. unitary?
  iv. positive definite?
(b) Prove that A^2 is hermitian.
(c) Prove that A^4 is positive definite.

(13) Let $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 \\ 2 & 4 & 2 & 2 \\ 2 & 4 & 2 & 4 \end{bmatrix}$. Which of the following is true? Circle the choices and give a brief but complete argument for your choice in the following assumptions.

(a) A is normal.
(b) A is unitary.
(c) A is hermitian.
(d) A has positive eigenvalues.
(e) A is unitarily diagonalizable.
(f) A is positive definite.

(14) Supppose that A is an $n \times n$ matrix and $A = PU$ is a polar decomposition. Show that A is normal iff P and U commute.

(15) Supppose that A is an $n \times n$ matrix and $A = PU$ is a polar decomposition. (Remember P is unique). Show that the eigenvalues of P are the singular values of A.

(16) If A is positive definite, so is U^*AU for any unitary U.

(17) Suppose that $B \in \mathcal{M}_n(\mathbb{C})$ is nonsingular. Show that B^*B is positive definite.

(18) Suppose A and B are positive definite. Show that their sum is also.

(19) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Show that A and B are positive definite, but AB is not.

(20) Problems on Perron's theorem in Horn's 2nd edition.

   (a) 8.0.P1-3
   (b) 8.1.P1-3
   (c) 8.2.P3-7, P13

(21) Suppose that A is a nonnegative matrix and that A^r is positive. Prove that A satisfies the properties in Perron's theorem: it has a unique eigenvalue of maximum modulus ρ, this eigenvalue has algebraic multiplicity 1 and positive left and right eigenvectors. What is the limit of $(A/\rho)^m$ as m goes to infinity?